Chapter 7: Techniques of Integration

Section 7.2: Trigonometric Integrals

What we will go over in this section…

1. $\int sin^m x\ cos^n x\ dx$
2. $\int tan^m x\ sec^n x\ dx$
3. $\int \sin mx \sin nx\ dx$

   $\int \sin mx \cos nx\ dx$

   $\int \cos mx \cos nx\ dx$

First, some derivatives you'll need to know…

$$(\sin x)' =$$

$$(\cos x)' =$$

$$(\tan x)' =$$

$$(\sec x)' =$$

Some Trig. Identities you need to know…

$$sin^2x + cos^2x = 1$$

$$\rightarrow sin^2x = 1 - cos^2x$$

$$\rightarrow cos^2x = 1 - sin^2x$$

$$tan^2x + 1 = sec^2x$$

$$\rightarrow tan^2x = sec^2x - 1$$

$$\rightarrow sec^2x = tan^2x + 1$$

$$1 + cot^2x = csc^2x$$

$$\rightarrow cot^2x = csc^2x - 1$$

$$\rightarrow csc^2x = 1 + cot^2x$$

Some Trig. Identities you need to know…

$$\sin(2x) = 2\sin x \cos x$$

$$\rightarrow \sin x \cos x = \frac{1}{2}\sin(2x)$$

Some Trig. Identities you need to know…

$$\cos(2x) = cos^2x - sin^2x$$

$$\cos(2x) = 2cos^2x - 1$$

$$\rightarrow cos^2x = \frac{1 + \cos(2x)}{2}$$

$$\cos(2x) = 1 - 2sin^2x$$

$$\rightarrow sin^2x = \frac{1 - \cos(2x)}{2}$$

Some Integrals you need to know…

$$\int \sin x \, dx =$$

$$\int \cos x \, dx =$$

Q: How would you do this integral?

$$\int \left(sin^4x + 3sin^3x - \sin x - 3\right) \cos x \;\; dx$$

1. Integrals of the form $\int sin^m x\, cos^n x\, dx$

Ex 1: Find $\int sin^4 x\, cos^5 x\, dx$

Similar Integrals:

$$\int sin^7 x\, cos^2 x\, dx$$

$$\int sin^3 x\, cos^9 x\, dx$$

1. Integrals of the form $\int sin^m x\, cos^n x\, dx$

<u>Ex 2</u>: Find $\int \; sin^2 x\, cos^4 x \;\; dx$

1. Integrals of the form $\int sin^m x \, cos^n x \, dx$

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

1. Integrals of the form $\int sin^m x \, cos^n x \, dx$

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

1. Integrals of the form $\int sin^m x\, cos^n x\, dx$

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

1. Integrals of the form $\int sin^m x \, cos^n x \, dx$

Examples from the book…

EXAMPLE 1 Evaluate $\int \cos^3 x \, dx$.

1. Integrals of the form $\int sin^m x \; cos^n x \; dx$

Examples from the book…

EXAMPLE 2 Find $\int \sin^5 x \cos^2 x \, dx$.

1. Integrals of the form $\int sin^m x\, cos^n x\, dx$

Examples from the book…

EXAMPLE 3 Evaluate $\int_0^{\pi} \sin^2 x\, dx$.

1. Integrals of the form $\int sin^m x \, cos^n x \, dx$

Examples from the book…

EXAMPLE 4 Find $\int \sin^4 x \, dx$.

Some More Integrals you need to know…

$$\int \tan x \, dx =$$

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$$\int \sec x \, dx =$$

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Q: How would you do these integrals?

$$\int (5tan^2 x + \tan x + 1)\, sec^2 x \;\; dx$$

$$\int (sec^5 x + sec^3 x + \sec x)\, sec\, x \tan x \;\; dx$$

2. Integrals of the form $\int tan^m x \, sec^n x \, dx$

Ex 3: Find $\int tan^8 x \, sec^6 x \, dx$

2. Integrals of the form $\int tan^m x \, sec^n x \, dx$

Ex 4: Find $\int tan^5 x \, sec^9 x \, dx$

Note: In the case of something like

$$\int tan^7 x \, sec^6 x \, dx$$

You can do the integral 2 different ways

2. Integrals of the form $\int tan^m x\ sec^n x\ dx$

Strategy for Evaluating $\int \tan^m x \sec^n x\, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x\, (\sec^2 x)^{k-1} \sec^2 x\, dx$$

$$= \int \tan^m x\, (1 + \tan^2 x)^{k-1} \sec^2 x\, dx$$

Then substitute $u = \tan x$.

2. Integrals of the form $\int tan^m x\ sec^n x\ dx$

Strategy for Evaluating $\int \tan^m x \sec^n x\, dx$

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x\ \sec x\ \tan x\, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x\ \sec x\ \tan x\, dx$$

Then substitute $u = \sec x$.

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

2. Integrals of the form $\int tan^m x\ sec^n x\ dx$

Examples from the book…

EXAMPLE 5 Evaluate $\int \tan^6 x \sec^4 x\, dx$.

2. Integrals of the form $\int tan^m x \; sec^n x \; dx$

Examples from the book…

EXAMPLE 6 Find $\int \tan^5\theta \, \sec^7\theta \, d\theta$.

2. Integrals of the form $\int tan^m x \, sec^n x \, dx$

<u>Ex 5 (book Ex. 7, pg. 483)</u>:

Find $\int tan^3 x \, dx$

2. Integrals of the form $\int tan^m x\ sec^n x\ dx$

Ex 6 (book Ex. 8, pg. 483):

Find $\int sec^3 x\ dx$

3. Integrals of the form $\int \sin mx \sin nx \, dx$, $\int \sin mx \cos nx \, dx$, or $\int \cos mx \cos nx \, dx$

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a)} \quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)} \quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)} \quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

3. Integrals of the form $\int \sin mx \sin nx \, dx$, $\int \sin mx \cos nx \, dx$, or $\int \cos mx \cos nx \, dx$

Ex 7: Find $\int \sin 8x \cos 5x \, dx$